Burnashev M. V.
Higher School of Modern Mathematics, MIPT
marat.burnashev@mail.ru

ON THE EXACT DECODING ERROR EXPONENT OF RANDOM CODING ON BSC

We consider the binary symmetric channel BSC(p) with crossover probability $0 < p < 1/2$, $q = 1 - p$. Its capacity $C(p) = \ln 2 - h(p)$, where $h(x) = -x \ln x - (1 - x) \ln(1 - x)$. For the given block length n and the transmission rate $R < C(p)$ a code $\mathcal{C}_n = \{\boldsymbol{x}_1^n, \ldots, \boldsymbol{x}_M^n\}$ of $M = e^{Rn}$ randomly chosen binary n-vectors is used. Denote by $P_{\rm e}(\mathcal{C}_n)$ the decoding error probability for the code $\mathcal{C}_n$ and by $P_{\rm e}(n, p, R) = \mathbf{E}_{\mathcal{C}_n} P_{\rm e}(\mathcal{C}_n, p, R)$ its expectation over $\mathcal{C}_n$. Define by $E_r(p, R) = \lim_{n\to\infty}\{-[\ln P_{\rm e}(n, p, R)]/n\}$ the decoding error exponent of that random coding.

Introduce critical rates $R_{\rm crit1}(p) = [(\sqrt{q} - \sqrt{p}) \ln(q/p)]/[2(\sqrt{q} + \sqrt{p})]$ and $R_{\rm crit}(p) = [(q - p) \ln(q/p)]/2 \geq R_{\rm crit1}(p)$ and denote $r_0 = 1/2 - R/\ln(q/p)$. We use the representation $R = \ln 2 - h(\delta_R)$, $0 \leq p \leq \delta_R \leq 1/2$. Next result supplements and sharpens results of [1,2] and it is based on the paper [3].

Theorem. For the function $E_r(p, R)$ the following formula holds

$$E_r(p,R) = \begin{cases} R - \ln 2 + \ln q + 2h(r_0) + r_0 \ln(p/q), & 0 \leq R \leq R_{\rm crit1}(p); \\ \ln 2 - 2\ln\left(\sqrt{q} + \sqrt{p}\right) - R, & R_{\rm crit1}(p) \leq R \leq R_{\rm crit}(p); \\ E_{\rm sp}(p,R), & R_{\rm crit}(p) \leq R \leq C(p), \end{cases} \qquad (1)$$

where $E_{\rm sp}(p, R) = \delta_R \ln(\delta_R/p) + (1 - \delta_R) \ln[(1 - \delta_R)/q]$.

References: [1] *Gallager R. G.* Information theory and reliable communication. Wiley, NY, 1968. [2] *Gallager R. G.* The Random Coding Bound Is Tight for the Average Code // IEEE Trans. Inform. Theory. 1973. V. 19. P. 244–246. [3] *Burnashev M. V.* On the Distribution of a Statistical Sum Related to the Binary Symmetric Channel // Problems of Information Transmission. V. 61, no. 1, pp. 11–7, 2025.