Spectral Sensitivity of Motifs and Block Phases in Dense Exponential Random Graph Models

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We consider dense exponential random graph models defined by motif densities. In the graphon limit, typical macroscopic states are described by maximizers of an energy-minus-entropy variational functional. The main question is which motif statistics are sensitive to a non-trivial block mode. For row-regular equal-block graphons, it is shown that for every tree T

$$t(T, W) = \left(\frac{\mu_1}{k}\right)^{|E(T)|},$$

where μ_1 is the common row sum of the connection matrix. Thus tree motifs depend only on the mean edge density and do not directly measure the block order parameter.

As a minimal model with an explicit order parameter, we study the motifs K_2, K_3, C_4 and the symmetric two-block parametrization $W_{q,m}$, where q is the mean density and m is the block contrast. In this reduction,

$$t(K_2, W_{q,m}) = q, \quad t(K_3, W_{q,m}) = q^3 + m^3, \quad t(C_4, W_{q,m}) = q^4 + m^4,$$

and the reduced free energy is

$$\Phi_\beta(q, m) = \beta_1 q + \beta_3(q^3 + m^3) + \beta_4(q^4 + m^4) - I(q, m).$$

Theorem. There exists a non-empty open parameter region $U_{\text{blk}} \subset \mathbb{R}^3$ such that, for every $\beta \in U_{\text{blk}}$, the maximum of Φ_β over $D = \{(q, m) : 0 \leq q \pm m \leq 1\}$ is attained at a point (q_*, m_*) with $m_* > 0$. Hence a non-trivial block phase appears in the reduced variational problem.