# FEYNMAN-KAC FORMULA FOR THE LAPLACE OPERATOR WITH A ZERO-RANGE POTENTIAL IN $\mathbb{R}^3$.

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Consider the self-adjoint operator $-\Delta$ defined on the domain $W_2^2(\mathbb{R}^3)$. Next, consider its restriction to the subspace $\mathcal{D}_0 = \{f \in W_2^2(\mathbb{R}^3) : f(\mathbf{0}) = 0\}$ The restricted operator has defect indices $(1,1)$.

Now, for each $k > 0$, we construct a self-adjoint extension $\mathcal{H}_k$ on the domain $\mathcal{D}(\mathcal{H}_k) = \mathcal{D}_0 \dotplus \mathcal{L}(\varphi_k)$, where $\mathcal{L}(\varphi_k)$ is the one-dimensional subspace spanned by the function $\varphi_k(\mathbf{x}) = \frac{e^{-kr}}{r} = \frac{e^{-k\|\mathbf{x}\|}}{\|\mathbf{x}\|}$.

On $\mathcal{D}(\mathcal{H}_k)$ we define the operator $\mathcal{H}_k$, by setting, for $f \in \mathcal{D}(\mathcal{H}_k)$ and $\mathbf{x} \neq 0$ $[\mathcal{H}_k f](\mathbf{x}) = -\Delta f(\mathbf{x})$. First the operator $\mathcal{H}_k$ appears in physical papers, one of the first mathematical paper where the properties of this operator were studied is [1].

For every $\varepsilon > 0$ define a function $L_\varepsilon(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3$, by setting $L_\varepsilon(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|}$ for $\|\mathbf{x}\| > \varepsilon$ and $L_\varepsilon(\mathbf{x}) = \frac{3\varepsilon^2 - \|\mathbf{x}\|^2}{2\varepsilon^3}$ for $\|\mathbf{x}\| \in [0, \varepsilon]$. Further, define a martingale $S_\varepsilon(\mathbf{x}, t)$, by

$$S_\varepsilon(\mathbf{x}, t) = \exp\left( \int_0^t \big( \frac{\nabla L_\varepsilon(\mathbf{w_x}(\tau))}{L_\varepsilon(\mathbf{w_x}(\tau))}, d\mathbf{w_x}(\tau) \big) - \frac{1}{2} \int_0^t \frac{\|\nabla L_\varepsilon(\mathbf{w_x}(\tau)\|^2}{\big(L_\varepsilon(\mathbf{w_x}(\tau)\big)^2} \, d\tau \right).$$

Using the martingale $S_\varepsilon(\mathbf{x}, t)$ we construct the operator family $P_k^t$, by setting for $f \in \mathcal{D}(\mathcal{H}_k)$

$$[P_k^t f](\mathbf{x}) = \frac{1}{\|\mathbf{x}\|} \lim_{\varepsilon \to 0} \mathbb{E} g(\mathbf{w_x}(t)) S_\varepsilon(\mathbf{x}, t),$$

where $g(\mathbf{x}) = f(\mathbf{x}) \cdot \|\mathbf{x}\|$.

**Theorem 1.** *For every $k > 0$ the family of operators $P_k^t$ is a semigroup with generator $-\frac{1}{2}\mathcal{H}_k$. This semigroup extends by continuity to a semigroup of self-adjoint bounded operators in $L_2(\mathbb{R}^3)$ of the form $P_k^t = e^{-\frac{t}{2}\mathcal{H}_k}$.*

## References

[1] F. A. Berezin, L. D. Faddeev, Remark on the Schrödinger equation with a singular potential, *Dokl. Akad. Nauk SSSR,* 137:5 (1961), Pages 1011–1014.

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