GENERALIZATION OF ONE RESULT OF V.V. SENATOV FOR THE DENSITY FUNCTION OF THE NORMALIZED SUM

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The introduction of a parameter to improve estimates in the Central Limit Theorem was proposed by H. Prawitz [1]. His result has been generalized by I.G. Shevtsova [4].

V.V. Senatov in [2] has obtained two expansions for the density function of the normalized sum from the Central Limit Theorem. In this expansions the last known moment of the initial distribution P is included in the main part of the expansion with some parameter. This allows to obtain the best explicit estimate of the remainder.

We generalize Senatov's results in the following theorem, which uses new asymptotic expansions [3] in the Central Limit Theorem as a basis. The proof of this result is based on the work [5].

Theorem. *Let independent identically random variables* $\xi,\ \xi_1,\ \xi_2, \ldots$ *with zero mean and unit variance have symmetric around zero probability distribution* P *with the finite moment* $\mathrm{M}\xi^{m+2}$ *of even order* $m+2 \geqslant 2$*, the real characteristic function* $f(t)$ *for which there is some positive number* $\nu > 0$ *that the function* $|f(t)|^{\nu}$ *is integrable on the whole real line. Then a the density function* $p_n(x)$ *of the normalized sum* $(\xi_1 + \cdots + \xi_n)\, n^{-1/2}$ *for* $n \geqslant \max\{\nu, m\}$*, all real* x *and* $0 \leqslant \lambda \leqslant 1$ *can be approximated as*

$$\left| p_n(x) - \phi(x) - \phi(x) \sum_{s=1}^{m/2} C_n^s \sum_{l=4s}^{m-4+4s} \frac{\Theta_{s,l}}{n^{l/2}} H_l(x) - \frac{\theta_{m+2}^{(\lambda)}}{n^{m/2}} \phi(x) H_l(x) \right| \lesssim \frac{\overline{\lambda}}{(m+2)!} \frac{\mathrm{M}\xi^{m+2}}{n^{m/2}} B_{m+2}\,,$$

where $\varphi(x)$ *is the probability density function of the standardized normal distribution,* B_{m+2} *is the moment of even order* $m+2$ *of the standard normal distribution divided by* $\sqrt{2\pi}$*,* $H_k(x) = (-1)^k \varphi^{(k)}(x)/\varphi(x)$ *is the Chebyshev-Hermite polynomials of degree* k *and* $\overline{\lambda} = \max\{\lambda;\ 1-\lambda\}$,

$$\theta_k = \sum_{j=0}^{[k/2]} \frac{(-1)^j}{2^j\, j!} \frac{\mathrm{M}\xi^{k-2j}}{(k-2j)!}, \qquad \theta_{m+2}^{(\lambda)} = \theta_{m+2} - \frac{(1-\lambda)}{(m+2)!} \mathrm{M}\xi^{m+2}, \qquad \Theta_{s,l} = \sum_{k_1+\cdots+k_s=l} \theta_{k_1} \cdots \theta_{k_s}\,,$$

where the sum is over all partitions of the integer $l = k_1 + \ldots + k_s$ *when* $k_j \geqslant 4,\ j = 1, \ldots, m-1$.

References

[1] H.Prawitz, "Noch einige Ungleichungen fur charakteristische Funktionen", *Scand. Actuar. J.*, **1**, 49-73, (1991).

[2] V.V. Senatov, "On the real accuracy of approximation in the central limit theorem. II", *Siberian Adv. Math.*, **27**, No. 2, 133-152, (2017).

[3] V.V. Senatov, V.N. Sobolev , "New forms of asymptotic expansions the central limit theorem", *Theory Probab. Appl.*, **57**, No. 1, 82-96, (2013).

[4] I.G. Shevtsova, "On the accuracy of the approximation of the complex exponent by the first terms of its Taylor expansion with applications", *J. Math. Anal. Appl.*, **418**, No. 1, 185-210 (2014).

[5] V.N. Sobolev, A. E. Condratenko "Generalization of one result of V.V. Senatov of characteristic functions of convolutions of probability", *XXXVI International Seminar on Stability Problems for Stochastic Models, Petrozavodsk, June 21 -- 25,* (2021).