On some Wald type identities for Fractional Brownian motions

Manuel L. Esquível and Nadezhda P. Krasii

May 17, 2022

Abstract

In 2017 A. N. Shiryaev asked us the following question: is there analogs of the Wald identities for the fractional Brownian motions? We believe so by reason of the following. There are series representation with Schauder functions—or similar functions—of all Brownian motions, usual and fractional; these series representations are obtained via the Haar basis and all have some remarkable characteristic: localisation of the random components. This characteristic lead us to believe that any property of the usual Brownian motion relying on this localisation feature may be extended, with adequate adaptations, to all fractional Brownian motions. This is the approach that subsumes the present work.

Let $(B_t^H)_{t\geq 0}$ the self similar process *fractional Brownian motion* for $0 < H < 1$.

$$B_t^H := C_H \left(\int_{-\infty}^0 \left((t-u)^{H-1/2} - (-u)^{H-1/2} \right) dB_u + \int_0^t (t-u)^{H-1/2} \, dB_u \right) ,$$

with,

$$C_H = \mathbb{E} \left[B_1^H \right]^{-\frac{1}{2}} \left(\int_{-\infty}^0 \left((t-u)^{H-1/2} - (-u)^{H-1/2} \right)^2 du + \frac{1}{2H} \right)^{-\frac{1}{2}} ,$$

where $(B_t)_{t\geq 0}$ is an usual Brownian process and the integrals must be given a special interpretation. We observe that there is a decomposition of this process, as the sum of two processes, $(1/C_H)B_t^H = R_t^H + M_t^H$ with,

$$R_t^H := \int_0^t (t-u)^{H-1/2} \, dB_u ,$$

and, *the long memory process* given by $M_0^H := 0$ and,

$$M_t^H := \int_0^{+\infty} \left((t+u)^{H-1/2} - u^{H-1/2} \right) d\widetilde{B}_u ,$$

$(\widetilde{B}_t)_{t\geq 0}$ being an independent copy of $(B_t)_{t\geq 0}$. We will treat the two processes separately. We have, with $\{\xi_0, \xi_{j,k} : j = 0, 1, \ldots, +\infty; k = 0, \ldots 2^j - 1\}$ in $\mathcal{H}$, a set of orthonormal random variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, that:

$$R_t^H = \xi_0 \frac{2t^{H+1/2}}{1+2H} + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \xi_{j,k} \varphi_{j,k}^H(t) \ ,$$

with an analog of the Schauder *little tent* functions defined by:

$$\varphi_{j,k}^H(t) := \frac{2^{\frac{j}{2}+1}}{1+2H} \left[\left(t - \frac{2k}{2^{j+1}} \right)^{H+\frac{1}{2}} \mathbb{1}_{\left[\frac{2k}{2^{j+1}}, \frac{2k+1}{2^{j+1}}\right[}(t) + \left(\frac{2k+2}{2^{j+1}} - t \right)^{H+\frac{1}{2}} \mathbb{1}_{\left[\frac{2k+1}{2^{j+1}}, \frac{2k+2}{2^{j+1}}\right[}(t) \right] ,$$

peaking with the value $2^{-(2H(j+1)+1)/2}/(1+2H)$ at the point $(2k+1)/2^{j+1}$. And, in the same way, we also have,

$$M_t^H = \xi_0 \frac{2\left(2^{H+1/2} - 2\right)}{1+2H} t^{H+1/2} + \sum_{j=0}^{+\infty} \sum_{k=0}^{2^j-1} \xi_{j,k} \psi_{j,k}^H(t) \ .$$

Let the filtration $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,1]}$ be defined, for $t > 0$, by:

$$\mathcal{G}_t := \sigma\left(\xi_0, \xi_{j,k}, j = 0, 1, \cdots + \infty;\ k = 0, \ldots 2^j - 1 : s \in \left[\frac{2k}{2^{j+1}}, \frac{2k+2}{2^{j+1}} \right[,\ s \leq t \right) \ ,$$

and let $\mathcal{G}_0 := \{\emptyset, \Omega\}$. Let τ be a stopping time with respect to $\mathbb{G}$.

Theorem 1 (Wald identity for fBm component R^H)**.** *Let τ be a stopping time with respect to $\mathbb{G}$ such that $\tau < 1$ almost surely. Then,*

$$\mathbb{E}\left[R_\tau^H\right] = \mathbb{E}\left[\xi_0 \frac{2}{1+2H} \tau^{H+1/2} \right] \ .$$

Theorem 2 (Wald identity for fBm component M^H)**.** *Let τ be a stopping time with respect to $\mathbb{G}$ such that $\tau < 1$ almost surely. Then,*

$$\mathbb{E}\left[M_\tau^H\right] = \mathbb{E}\left[\xi_0 \frac{2\left(2^{H+1/2} - 2\right)}{1+2H} \tau^{H+1/2} \right] \ .$$

References

[1] Pavel Yaskov. A maximal inequality for fractional brownian motions. *Journal of Mathematical Analysis and Applications*, 472(1):11–21, 2019.