Pricing double barrier options under Lévy processes of unbounded variation

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Keywords: Wiener-Hopf factorization, Lévy processes, Numerical methods, Option pricing

The most popular path-dependent options are barrier options, which include double barrier options. Let a stochastic process S_t be a model for the chosen stock price dynamics. Recall that a double barrier option on the stock is a contract which pays the specified amount $G(S_T)$ at the terminal date T, provided during its lifetime, the price of the stock does not cross specified constant barriers D from above and U from below. When at least one of the barriers is crossed, the option expires worthless, or the option owner is entitled to some *rebate*.

From a probabilistic viewpoint, one can express double barrier option prices in terms of conditional expectation on a payoff function that depends on the underlying stochastic process and its extrema. Notice that the known results on pricing double barrier options are rather limited. In analytical terms, the option pricing problem under consideration leads to a matrix Wiener-Hopf factorization (see details in [3]), which is not analytically available yet. To treat the problem in general case numerically, one should apply the Laplace transform (or the Carr's randomization), then solve two coupled complex integrodifferential equations that require complicated approximate formulas for the Wiener-Hopf factors. The overview of the existing numerical methods can be found in [2, 5, 6, 8, 11, 9, 12, 10]. Therefore, pricing double barrier options in exponential Lévy models remains a computational challenge.

In the paper [10], the author suggested a new approach for pricing exotic options with a payoff depending on the infimum and supremum of Lévy processes at expiry. The method suggested makes it easy to implement such a sophisticated tool as the Wiener-Hopf factorization for general Lévy models with jumps of finite variation. The goal of the current paper is to extend the approach from [10] to pure non-Gaussian Lévy processes with jumps of unbounded variation. The main advantage of the method is applying semi-explicit Wiener-Hopf factorization formulas.

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see, e.g., [4]). A Lévy model may have a Gaussian component and pure jump component. A Lévy process X_t can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$ (we confine ourselves to the one-dimensional case).

The Lévy-Khintchine formula gives the characteristic exponent of a pure non-Gaussian Lévy process:

$$-i\gamma\xi + \int_{\mathbf{R}} (1 - e^{i\xi x} + i\xi x \mathbf{1}_{[-1,1]}(x))\Pi(dx), \tag{1}$$

where $\gamma \in \mathbf{R}$ is the drift, $\mathbf{1}_A$ is the indicator function of the set A, and the Lévy measure $\Pi(dx)$

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satisfies $\int_{\mathbf{R}} \min\{1, x^2\}\Pi(dx) < +\infty$. If the condition

$$\int_{\mathbf{R}} \min\{1, |x|\}\Pi(dx) < +\infty. \tag{2}$$

does not hold, then the Lévy process X_t is of unbounded variation.

Let T, K, D, U be the maturity, strike, the lower barrier, and the upper barrier, and the stock price $S_t = De^{X_t}$ be an exponential Lévy process under a chosen risk-neutral measure which is pure non-Gaussian with jumps of unbounded variation. Without loss of generality, we confine ourselves to a double barrier put option. Set the riskless rate and the dividend rate equal to r and d, respectively. We consider an approach to pricing continuously monitored double barrier put options without rebate under a Lévy process with the characteristic exponent (1) that does not satisfies (2).

Let us introduce $h = \ln U/D$. Then the payoff at maturity is $\mathbf{1}_{(0,h)}(X_T)G(X_T)$, where $G(x) = (K - De^x)_+$, and the no-arbitrage price of the double barrier put option at the beginning of a period under consideration ($t = 0$) and $X_t = x$ with $x \in (0, h)$ given by

$$V(T, x) = E^x\left[e^{-rT}\mathbf{1}_{\underline{X}_T>0}\mathbf{1}_{\overline{X}_T<h}G(X_T)\right], \tag{3}$$

where T is the final date, $\underline{X}_t = \inf_{0\le s\le t} X_t$ and $\overline{X}_t = \sup_{0\le s\le t} X_t$ are the infimum and the supremum of the process X_t, respectively. The short-hand notation $E^x[\cdot]$ means that we take the expectation conditioned on the event $X_0 = \underline{X}_0 = \overline{X}_0 = x$.

Theorem 1. *Let N be a sufficiently large natural number. Set $q = T/N$, $v_0(q, x) = G(x)\mathbf{1}_{(0,h)}(x)$, and for $n = 1, 2, \dots$ define*

$$v_n(q, x) = E^x\left[\frac{v_{n-1}(q, X_{T_{q+r}})}{(1 + r/q)}\mathbf{1}_{\underline{X}_{T_{q+r}}>0}\mathbf{1}_{\overline{X}_{T_{q+r}}<h}\right], \tag{4}$$

where the random time $T_{q+r} \sim \mathrm{Exp}\,(q + r)$.

For a fixed x, $v_N(N/T, x)$ converges to $V(T, x)$ as $N \to +\infty$.

We prove Theorem 1 by using Laplace transform techniques and Post-Widder approximate formula. Thus, we need a method to compute efficiently the right hand side of (4).

The new approach to calculating (4) requires the following steps. The key idea behind the method is to represent the process X_t as the sum of spectrally positive jumps X^+_t with a non-negative drift and spectrally negative jumps X^-_t with a non-positive drift: $X_t = X^+_t + X^-_t$.

Let $X^{+,1}_t$ and $X^{+,2}_t$ be Lévy processes with the same characteristic exponent, i.e. $X^{+,1}_t \sim X^+_t$ and $X^{+,2}_t \sim X^+_t$. Due to the property of increments of a Lévy process to be stationary independent and characteristics of the supremum and infimum processes, we conclude that X_t and $Y_t(= \underline{X}^{+,1}_{t/2} + \overline{X}^{+,1}_{t/2} + \overline{X}^-_t + \underline{X}^-_t + \underline{X}^{+,2}_{t/2} + \overline{X}^{+,2}_{t/2})$ are identically distributed.

Let a natural number N be sufficiently large and $q = N/T$. Since the randomized time T_{q+r} converges in mean square sense to 0 as $N \to +\infty$, we may approximate $X_{T_{q+r}}$ in (4) with $Y_{T_{q+r}}$. Notice that $T_{q+r}/2$ is also an exponentially distributed random variable but with the intensity parameter equal to $2(q + r)$. We show that $X^+_{T_{2(q+r)}}$ and $X^-_{T_{q+r}}$ admit semi-explicit Wiener-Hopf factorizations.

Theorem 2. *Let $q > 0$ be sufficiently large. Then for a fixed $\xi \in \mathbf{R}$*

$$E[e^{i\xi X(T_q)}] - E[e^{i\xi Y(T_q)}] \sim O(q^{-2}) \text{ as } q \to +\infty.$$

Based on Theorem 2 we suggest the following numerical procedure for computation of (4).

Theorem 3. *Let a natural number N be sufficiently large and $q = N/T$. Introduce the following operators:*

$$\begin{aligned}
\mathcal{E}_+^+ u(x) &= E[u(x + \overline{X}^+_{T_{q+r/2}})],\ \mathcal{E}_-^+ u(x) = E[u(x + \overline{X}^-_{T_{q+r}})]; \\
\mathcal{E}_+^- u(x) &= E^x[u(\underline{X}^+_{T_{q+r/2}})],\ \mathcal{E}_-^- u(x) = E^x[u(\underline{X}^-_{T_{q+r}})].
\end{aligned}$$

One may approximate $v_n(q, x)$ in (4) as follows:

$$v_n(q, x) = \frac{\mathbf{1}_{(0,h)}(x)}{(1 + r/q)} \mathcal{E}_-^+ \mathbf{1}_{(0,h)} \mathcal{E}_+^+ \mathcal{E}_-^+ \mathbf{1}_{(0,h)} \mathcal{E}_-^- \mathcal{E}_+^- \mathbf{1}_{(0,h)} \mathcal{E}_+^+ v_{n-1}(q, x) + O(q^{-2}) \text{ as } q \to +\infty.$$

The operators $\mathcal{E}_+^+$, $\mathcal{E}_-^+$, $\mathcal{E}_+^-$ and $\mathcal{E}_-^-$ can be efficiently implemented by using the Fast Fourier Transform (FFT) for real-valued functions (see e.g. [11]).

In the paper, we suggested a new approach for pricing exotic options with a payoff depending on the infimum and supremum of Lévy processes at expiry. The method suggested makes it easy to implement such a sophisticated tool as the Wiener-Hopf factorization for general Lévy models with jumps of unbounded variation.

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