A new approach to the expansion of iterated Stratonovich stochastic integrals based on multiple Fourier–Legendre series and multiple trigonometric Fourier series: multiplicities 1 to 5 and beyond

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The work is devoted to a new approach to the expansion of iterated Stratonovich stochastic integrals with respect to the components of a multidimensional Wiener process. This approach is based on multiple Fourier–Legendre series as well as multiple trigonometric Fourier series. The theorem on the mean-square convergent expansion for the iterated Stratonovich stochastic integrals of arbitrary multiplicity is formulated and proved under the condition of convergence of trace series. This condition has been verified for integrals of multiplicities from 2 to 5 and complete orthonormal systems of Legendre polynomials and trigonometric functions in Hilbert space. The Hu–Meyer formula [1] and multiple Wiener–Itô stochastic integral [2] were used in the proof of the mentioned theorem. The rate of mean-square convergence of the obtained expansions is found. The results of the article can be applied to the numerical integration of Itô stochastic differential equations with non-commutative noise in the framework of the approach based on the Taylor–Stratonovich expansion.

Let $(\Omega,\ \mathrm{F},\ \mathsf{P})$ be a complete probability space, let $\{\mathrm{F}_\tau, \tau\in[0,T]\}$ be a nondecreasing right-continous family of σ-algebras of F, and let $\mathbf{W}_\tau$ be a standard m-dimensional Wiener stochastic process, which is F_τ-measurable for any $\tau\in[0,T]$. We assume that the components $\mathbf{W}_\tau^{(i)}$ $(i=1,\ldots,m)$ of this process are independent.

Consider the unordered set $\{1,2,\ldots,k\}$ and separate it into two parts: the first part consists of r unordered pairs (sequence order of these pairs is also unimportant) and the second one consists of the remaining $k-2r$ numbers. So, we have

$$(\{\underbrace{\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}}_{\text{part 1}}\},\{\underbrace{q_1,\ldots,q_{k-2r}}_{\text{part 2}}\}), \tag{1}$$

where $\{g_1,g_2,\ldots,g_{2r-1},g_{2r},q_1,\ldots,q_{k-2r}\}=\{1,2,\ldots,k\}$, braces mean an unordered set, and parentheses mean an ordered set.

Consider the Fourier coefficient

$$C_{j_k\ldots j_1}=\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_k \tag{2}$$

corresponding to the function $K(t_1,\ldots,t_k)=\psi_1(t_1)\ldots\psi_k(t_k)\mathbf{1}_{\{t_1<\ldots<t_k\}},\ t_1,\ldots,t_k\in[t,T],\ k\ge 2$ and $K(t_1)=\psi_1(t_1),\ t_1\in[t,T]$. Here $\psi_1(\tau),\ldots,\psi_k(\tau):[t,T]\to\mathbb{R}$ are nonrandom functions, $\{\phi_j(x)\}_{j=0}^{\infty}$ is a complete orthonormal system of functions in the space $L_2([t,T])$, $j_1,\ldots,j_k=0,1,\ldots,$ $\mathbf{1}_A$ denotes the indicator of the set A. At that we suppose $\phi_0(x)=1/\sqrt{T-t}$. Here and further, we assume that $0\le t<T<\infty$.

Denote

$$C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}\Big|_{(j_lj_l)\frown(\cdot)}\stackrel{\mathsf{def}}{=}\int\limits_t^T\psi_k(t_k)\phi_{j_k}(t_k)\ldots\int\limits_t^{t_{l+2}}\psi_{l+1}(t_{l+1})\phi_{j_{l+1}}(t_{l+1})\int\limits_t^{t_{l+1}}\psi_l(t_l)\psi_{l-1}(t_l)\times$$

$$\times\int\limits_t^{t_l}\psi_{l-2}(t_{l-2})\phi_{j_{l-2}}(t_{l-2})\ldots\int\limits_t^{t_2}\psi_1(t_1)\phi_{j_1}(t_1)dt_1\ldots dt_{l-2}dt_lt_{l+1}\ldots dt_k, \tag{3}$$

i.e. $C_{j_k\ldots j_{l+1}j_lj_lj_{l-2}\ldots j_1}|_{(j_lj_l)\frown(\cdot)}$ is again the Fourier coefficient of type $C_{j_k\ldots j_1}$ but with a new shorter multi-index $j_k\ldots j_{l+1}0j_{l-2}\ldots j_1$ and new weight functions $\psi_1(\tau),\ \ldots,\ \psi_{l-2}(\tau),\ \sqrt{T-t}\psi_{l-1}(\tau)\psi_l(\tau)$,

$\psi_{l+1}(\tau),\ \ldots,\ \psi_k(\tau)$ (also we suppose that $\{l,l-1\}$ is one of the pairs $\{g_1,g_2\},\ldots,\{g_{2r-1},g_{2r}\}$ from the relation (1).

Denote

$$\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\stackrel{\sf def}{=}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}},$$

$$S_l\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\frac{1}{2}\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}\sum_{j_{g_{2r-1}}=p+1}^{\infty}\ \sum_{j_{g_{2r-3}}=p+1}^{\infty}\ldots$$

$$\ldots\sum_{j_{g_{2l+1}}=p+1}^{\infty}\ \sum_{j_{g_{2l-3}}=p+1}^{\infty}\ldots\sum_{j_{g_3}=p+1}^{\infty}\ \sum_{j_{g_1}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}.$$

Note that the operation S_l $(l=1,2,\ldots,r)$ acts on the value $\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$ as follows: S_l multiplies $\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$ by $\mathbf{1}_{\{g_{2l}=g_{2l-1}+1\}}/2$, removes the summation with respect to $j_{g_{2l-1}}$, and replaces $C_{j_k\ldots j_1}|_{j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$ with $C_{j_k\ldots j_1}|_{(j_{g_{2l}}j_{g_{2l-1}})\curvearrowright(\cdot),j_{g_1}=j_{g_2},\ldots,j_{g_{2r-1}}=j_{g_{2r}}}$. At that we write $C_{j_k\ldots j_1}|_{(j_{g_1}j_{g_2})\curvearrowright(\cdot),j_{g_1}=j_{g_2}}=C_{j_k\ldots j_1}|_{(j_{g_1}j_{g_1})\curvearrowright(\cdot),j_{g_1}=j_{g_2}}$.

The action of superposition S_lS_m on $\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}$ is obvious. For example, for $r=3$

$$S_3S_1\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_5,g_6}\right\}=$$

$$=\frac{1}{2^2}\mathbf{1}_{\{g_6=g_5+1\}}\mathbf{1}_{\{g_2=g_1+1\}}\sum_{j_{g_3}=p+1}^{\infty}C_{j_k\ldots j_1}\Biggl|_{(j_{g_2}j_{g_1})\curvearrowright(\cdot)(j_{g_6}j_{g_5})\curvearrowright(\cdot),j_{g_1}=j_{g_2},j_{g_3}=j_{g_4},j_{g_5}=j_{g_6}}.$$

Theorem 1 [3] (Sect. 2.10). *Assume that the continuously differentiable functions $\psi_l(\tau)$ $(l=1,\ldots,k)$ and the complete orthonormal system $\{\phi_j(x)\}_{j=0}^{\infty}$ of continuous functions $(\phi_0(x)=1/\sqrt{T-t})$ in the space $L_2([t,T])$ are such that the following conditions are satisfied:*

1. *The equality*

$$\frac{1}{2}\int\limits_t^s\Phi_1(\tau)\Phi_2(\tau)d\tau=\sum_{j=0}^{\infty}\int\limits_t^s\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau \qquad (4)$$

holds for all $s\in(t,T]$, where the nonrandom functions $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable on $[t,T]$ and the series on the right-hand side of (4) converges absolutely.

2. *The estimates*

$$\left|\int\limits_t^s\phi_j(\tau)\Phi_1(\tau)d\tau\right|+\left|\int\limits_s^T\phi_j(\tau)\Phi_2(\tau)d\tau\right|\le\frac{\Psi_1(s)}{j^{1/2+\alpha}},\quad\left|\sum_{j=p+1}^{\infty}\int\limits_t^s\Phi_2(\tau)\phi_j(\tau)\int\limits_t^{\tau}\Phi_1(\theta)\phi_j(\theta)d\theta d\tau\right|\le\frac{\Psi_2(s)}{p^{\alpha}}$$

hold for all $s\in(t,T)$ and for some $\alpha>0$, where $\Phi_1(\tau)$, $\Phi_2(\tau)$ are continuously differentiable nonrandom functions on $[t,T]$, $j,p\in\mathbb{N}$, and $\Psi_1(\tau)\in L_2([t,T])$, $\Psi_2(\tau)\in L_1([t,T])$.

3. *The condition*

$$\lim_{p\to\infty}\sum_{\substack{j_1,\ldots,j_q,\ldots,j_k=0\\ q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}}^{p}\left(S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\right)^2=0$$

holds for all possible $g_1,g_2,\ldots,g_{2r-1},g_{2r}$ (see (1)) and $l_1,l_2,\ldots,l_d$ such that $l_1,l_2,\ldots,l_d\in\{1,2,\ldots,r\}$, $l_1>l_2>\ldots>l_d$, $d=0,1,2,\ldots,r-1$, where $r=1,2,\ldots,[k/2]$ and

$$S_{l_1}S_{l_2}\ldots S_{l_d}\left\{\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\right\}\stackrel{\sf def}{=}\bar C^{(p)}_{j_k\ldots j_q\ldots j_1}\Biggl|_{q\ne g_1,g_2,\ldots,g_{2r-1},g_{2r}}\quad \textit{for}\quad d=0.$$

Then, for the iterated Stratonovich stochastic integral of arbitrary multiplicity k

$$J^{\circ}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\int\limits_t^T\psi_k(t_k)\ldots\int\limits_t^{t_2}\psi_1(t_1)\circ d\mathbf{W}_{t_1}^{(i_1)}\ldots\circ d\mathbf{W}_{t_k}^{(i_k)} \tag{5}$$

the following expansion

$$J^{\circ}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)} \tag{6}$$

that converges in the mean-square sense is valid, where $C_{j_k\ldots j_1}$ *is the Fourier coefficient defined by* (2), l.i.m. *is a limit in the mean-square sense,* $i_1,\ldots,i_k=0,1,\ldots,m,$

$$\zeta_j^{(i)}=\int\limits_t^T\phi_j(\tau)d\mathbf{W}_{\tau}^{(i)}$$

are independent standard Gaussian random variables for various i *or* j *(in the case when* $i\neq 0$*),* $d\mathbf{W}_{\tau}^{(i)}$ *and* $\circ d\mathbf{W}_{\tau}^{(i)}$ $(i=0,1,\ldots,m)$ *are Itô and Stratonovich differentials, respectively;* $\mathbf{W}_{\tau}^{(0)}\stackrel{\text{def}}{=}\tau$.

The following theorem is proved [3] on the base of Theorem 1.

Theorem 2 [3] (Sect. 2.11–2.15). *Suppose that* $\{\phi_j(x)\}_{j=0}^{\infty}$ *is a complete orthonormal system of Legendre polynomials or trigonometric functions in the space* $L_2([t,T])$*. Furthermore, let* $\psi_1(\tau),\ldots,\psi_5(\tau)$ *are continuously differentiable nonrandom functions on* $[t,T]$*. Then, for the iterated Stratonovich stochastic integral* $J^{\circ}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}$ $(k=3,4,5)$ *defined by* (5) *the following relations*

$$J^{\circ}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}=\underset{p\to\infty}{\text{l.i.m.}}\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}, \tag{7}$$

$$\mathsf{M}\left(J^{\circ}[\psi^{(k)}]_{T,t}^{(i_1\ldots i_k)}-\sum_{j_1,\ldots,j_k=0}^{p}C_{j_k\ldots j_1}\zeta_{j_1}^{(i_1)}\ldots\zeta_{j_k}^{(i_k)}\right)^2\le\frac{C}{p^{1-\varepsilon}} \tag{8}$$

are fulfilled, where $i_1,\ldots,i_k=0,1,\ldots,m$ *in* (7) *and* $i_1,\ldots,i_k=1,\ldots,m$ *in* (8)*, constant* C *is independent of* p*,* ε *is an arbitrary small positive real number for the case of complete orthonormal system of Legendre polynomials in the space* $L_2([t,T])$ $(k=4,5)$ *and* $\varepsilon=0$ *for the case of complete orthonormal system of trigonometric functions in the space* $L_2([t,T])$ $(k=4,5)$ *or for the cases* $k=2,3$*,* $C_{j_k\ldots j_1}$ *is the Fourier coefficient defined by* (2)*, another notations are the same as in Theorem* 1*.*

Note that (7), (8) are also valid for the case $k=1$ with $\varepsilon=0$ (see Theorems 1.1, 1.16 and Remark 1.7 [3]). The case $k=2$ of Theorem 2 as well as some narrow special cases of Theorem 2 for $k=3,4$ were considered earlier in [3] (Chapter 2).

An expansion similar to (6) (without estimating the rate of mean-square convergence) was obtained in [4] using a different approach.

References

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