Kac-Ornstein-Uhlenbeck processes

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Let $\varepsilon = \varepsilon(t) \in \{0, 1\},\ t \geq 0$, be a two-state continuous-time Markov chain switching with alternating intensities $\lambda_0\ \lambda_1 > 0$. Let $(a_0, a_1),\ (\gamma_0, \gamma_1)$ be two pairs of real numbers.
A new class of piecewise deterministic processes is studied, which are defined by the following integral equation

$$X(t) = x + \int_0^t \Big(a_\varepsilon(s) - \gamma_{\varepsilon(s)} X(s)\Big)\,\mathrm{d}s. \tag{1}$$

We call the solution of this equation the Kac-Ornstein-Uhlenbeck process, since under the standard Kac's scaling, i.e., if $\gamma_0, \gamma_1,\ \lambda_0, \lambda_1 \to \infty,\ \gamma_0^2/\lambda_0, \gamma_1^2/\lambda_1 \to \sigma^2$, then the telegraph process $\Gamma(t) = \int_0^t \gamma_{\varepsilon(s)}\mathrm{d}s$ converges to the Brownian motion $W = W_t$, [3], and the process $X = X(t)$ converges to an Ornstein-Uhlenbeck process $\bar{X}$ satisfying the Langevin equation

$$\bar{X}(t) = x + at - \int_0^t \bar{X}(s)\mathrm{d}W_s, \tag{2}$$

(if additionally $a_0, a_1 \to a$). The trajectories of the process $X = X(t)$ are formed by two deterministic patterns

$$\phi_0(t, x) = \mathrm{e}^{-\gamma_0 t}\left(x + a_0 \int_0^t \mathrm{e}^{\gamma_0 s}\mathrm{d}s\right) = \rho_0 + (x - \rho_0)\mathrm{e}^{-\gamma_0 t},$$

$$\phi_1(t, x) = \mathrm{e}^{-\gamma_1 t}\left(x + a_1 \int_0^t \mathrm{e}^{\gamma_1 s}\mathrm{d}s\right) = \rho_1 + (x - \rho_1)\mathrm{e}^{-\gamma_1 t},$$

$\rho_0 = a_0/\gamma_0 \neq \rho_1 = a_1/\gamma_1,\ t \geq 0$, continuously switching from one to the other at random times $\tau_n,\ n \geq 1,\ \tau_0 = 0$.
If the parameters ρ_0 and ρ_1 coincide, $a_0/\gamma_0 = a_1/\gamma_1 =: \rho$, then $X = X(t)$ reduces to the geometric telegraph process:

$$X(t) = \mathrm{e}^{-\Gamma(t)}\left(x + \rho \int_0^t \gamma_{\varepsilon(s)}\mathrm{e}^{\Gamma(s)}\mathrm{d}s\right) = \rho + (x - \rho)\exp(-\Gamma(t)). \tag{3}$$

The properties of such a process are well studied, see [3, 4, 7]. Note that in this case the process X is time-homogeneous in the sense of (2.13), [6].with a rectifying diffeomorphism $\Phi(x) = \log|x - \rho|$. In this case, the distribution of $X(t)$ is determined by the distribution of the telegraph process $\Gamma(t)$. In what follows, we assume $\rho_0 < \rho_1$.

1. Stationary distributions. Let $\gamma_0,\ \gamma_1 > 0$.
Since after almost surely finite transition time the paths of X fall into $(\rho_0,\ \rho_1)$ and remain inside this interval, the invariant measure $\vec{\mu}$ is supported on $[\rho_0,\ \rho_1]$. Let $\alpha_0 = \lambda_0/\gamma_0, \alpha_1 = \lambda_1/\gamma_1,\ \alpha_0, \alpha_1 > 0$.

Theorem 1. *The unique invariant probability measure for $\Xi = (X(t),\ \varepsilon(t)),\ t \geq 0$, has the form of a Beta distribution determined by the probability density functions $\vec{\pi} = (\pi_0(x),\ \pi_1(x)),\ \rho_0 < x < \rho_1$,*

$$\pi_0(x) = \frac{\lambda_1}{\lambda_0 + \lambda_1}(\rho_1 - \rho_0)^{-1} B(\alpha_0, 1 + \alpha_1)^{-1} \cdot \xi_0(x)^{-1+\alpha_0}\xi_1(x)^{\alpha_1},$$

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$$\pi_1(x) = \frac{\lambda_0}{\lambda_0+\lambda_1}(\rho_1-\rho_0)^{-1}B(1+\alpha_0,\alpha_1)^{-1}\cdot\xi_0(x)^{\alpha_0}\xi_1(x)^{-1+\alpha_1}.$$

Here $\xi_0(x) = \frac{x-\rho_0}{\rho_1-\rho_0}$, $\xi_1(x) = 1-\xi_0(x) = \frac{\rho_1-x}{\rho_1-\rho_0}$, *and* $B(\alpha_0,\ \alpha_1)$ *is the Euler beta-function.*
In the attraction-repulsion case $\gamma_0\cdot\gamma_1<0$, the invariant distribution exists only if

$$\alpha_0+\alpha_1<0.$$

Theorem 2. *If* $\gamma_0>0>\gamma_1$, *then the invariant probability measure for the Kac-Ornstein-Uhlenbeck process* X *is supported on the half-line* $x<\rho_0=a_0/\gamma_0$, *and the probability density functions* π_0 *and* π_1, *are given by*

$$\pi_0(x) = \frac{\lambda_1}{\lambda_0+\lambda_1}(\rho_1-\rho_0)^{-1}B(-\alpha_0-\alpha_1,\alpha_0)^{-1}[-\xi_0(x)]^{-1+\alpha_0}\xi_1(x)^{\alpha_1}\mathbf{1}_{\{\mathbf{x}<\rho_\mathbf{0}\}},$$

$$\pi_1(x) = \frac{-\alpha_1\gamma_0}{\lambda_0+\lambda_1}(\rho_1-\rho_0)^{-1}B(-\alpha_0-\alpha_1,\alpha_0)^{-1}[-\xi_0(x)]^{\alpha_0}\xi_1(x)^{-1+\alpha_1}\mathbf{1}_{\{\mathbf{x}<\rho_\mathbf{0}\}}.$$

In the case $\gamma_0<0<\gamma_1$, *the invariant measure is supported on the upper half-line* $x>\rho_1$ *and looks similarly.*

2. Exponential functionals Consider the exponential functional of the form

$$\mathcal{G}_{\gamma,a} = \int_0^\infty \mathrm{e}^{-\Gamma(t)}a_{\varepsilon(t)}\mathrm{d}t = \lim_{T\to\infty}\int_0^T \mathrm{e}^{-\Gamma(t)}a_{\varepsilon(t)}\mathrm{d}t, \qquad a.s. \tag{4}$$

It is useful to note that the stationarity of Markov-modulated Ornstein-Uhlenbeck process and the distribution of the corresponding exponential functional are closely related. This problem has been studied in detail, see e.g. [1, 2]. It is interesting to note that this more general setting can be transformed in the particular case of the Kac-Ornstein-Uhlenbeck process driven by a pair of telegraph processes, see [5].

References

1. Behme A. and Sideris A. (2020) Exponential functionals of Markov additive process. *Electron. J. Probab.* **25**, no. 37, 1-25.

2. Behme A. and Sideris A. (2022) Markov-modulated generalized Ornstein-Uhlenbeck processes and an application in risk theory. *Bernoulli* **2**8(2): 1309-1339

3. López O. and Ratanov N. (2014) On the asymmetric telegraph processes. *J. Appl. Probab.*, **51**(2), 569–589.

4. Ratanov N. (2007) A jump telegraph model for option pricing. *Quant. Fin.* **7**, 575–583.

5. Ratanov N. (2022) Kac-Ornstein-Uhlenbeck processes: stationary distributions and exponential functionals. To appear in *Methodol. Comput. Appl. Probab.*

6. Ratanov N, Di Crescenzo A. and Martinucci B. (2019) Piecewise deterministic processes following two alternating patterns. *J. Appl. Prob.* **56**, 1006–1019.

7. Kolesnik A. D. and Ratanov N. (2013) *Telegraph Processes and Option Pricing.* Springer: Heidelberg.