Decompositions of Finitely Additive Markov Chains and Asymptotics of their Components*

Alexander Zhdanok[1], Anna Khuruma[2]

[1]Institute for Information Transmission Problems RAS, Moscow, Russia; zhdanok@iitp.ru
[2]Tuvan State University, Kyzyl, Russia; huruma@list.ru

We consider Markov chains (MC) defined by the transition probability (kernel) which is finitely additive. Such Markov chains were constructed by S. Ramakrishnan [1] within the concepts and symbolism of the game theory. We study such MCs using the operator approach. In our work, the state space (phase space) of the MC has any cardinality, and the sigma-algebra is discrete.This construction of the phase space allows us to decompose the Markov kernel (and the Markov operators it generates) into the sum of two components - countably additive and purely finitely additive. Some asymptotic regularities of such MCs were revealed. In our previous papers [2],[3] and [4] we also study Markov sequences of finitely additive measures. However, they are generated by operators with a countably additive kernel extended to the space of finitely additive measures.

1 Definitions, Notation and Some Constructions

Let X be an arbitrary infinite set and Σ the sigma-algebra of its subsets. Let $B(X, \Sigma)$ denote the Banach space of bounded Σ -measurable functions $f : X \to R$ with sup-norm.

We also consider Banach spaces of bounded measures $\mu : \Sigma \to R$, $ba(X, \Sigma)$ is the space of finitely additive measures, $ca(X, \Sigma)$ is the space of countably additive measures, $pfa(X, \Sigma)$ is the space of purely finitely additive measures.

The Yosida-Hewitt decomposition is known: Any finitely additive measure μ can be uniquely decomposed into the sum $\mu = \mu_{ca} + \mu_{pfa}$, where μ_{ca} is countably additive and μ_{pfa} is a purely finitely additive measure.

We denote the sets of non-negative measures: $V_{ba} = \{\mu \in ba(X, \Sigma) : \mu(X) \leq 1\}, V_{ca} = \{\mu \in ca(X, \Sigma) : \mu(X) \leq 1\}, V_{pfa} = \{\mu \in pfa(X, \Sigma) : \mu(X) \leq 1\}$.

Measures from these sets will be called probabilistic if $\mu(X) = 1$.

We also denote by S_{ba}, S_{ca}, S_{pfa} the sets of all probability measures in V_{ba}, V_{ca}, V_{pfa}, respectively.

Definition 1 *The finitely additive Markov chains (MC) on a measurable space* (X, Σ) *are given by their transition function (probability kernel)* $P(x, E), x \in X, E \in \Sigma$, *under the conditions:*

1) $0 \leq P(x, E) \leq 1, \forall x \in X, \forall E \in \Sigma$;
2) $P(\cdot, E) \in B(X, \Sigma), \forall E \in \Sigma$;
3) $P(x, \cdot) \in ba(X, \Sigma), \forall x \in X$;
4) $P(x, X) = 1, \forall x \in X$.

We emphasize that the transition function of the classical Markov chain is a countably additive measure in the second argument.

*This research was funded by the Russian Foundation of Basic Research, project number 20-01-00575-a.

The transition function generates Markov linear integral operator:
$A : ba(X,\Sigma) \to ba(X,\Sigma), (A\mu)(E) = A\mu(E) = \int\limits_X P(x,E)\mu(dx), \forall \mu \in ba(X,\Sigma), \forall E \in \Sigma.$

Let the initial measure be $\mu^1 \in S_{ba}$. Then the iterative sequence of finitely additive probability measures $\mu^{n+1} = A\mu^n \in S_{ba}, n \in N$, is usually identified with the Markov chain. We will call $\{\mu^n\}$ a Markov sequence of measures.

Definition 2 *A topological space (X,τ) is called discrete if all its subsets are simultaneously open and closed (clopen), that is, the topology $\tau = 2^X$ - the set of all subsets of the set X.*

If a topological space is discrete, then, obviously, its Borel sigma-algebra $\mathfrak{B} = \tau = 2^X$. Such a sigma-algebra in X is also called discrete. We will denote it by Σ_d.

Proposition 1 *Let an arbitrary discrete space (X,Σ_d) be given. Any Markov finitely additive kernel $P(x,E)$ on (X,Σ_d) is uniquely representable as the sum of a sub-Markov countably additive kernel $P_{ca}(x,E)$ and a sub-Markov purely finitely additive kernel $P_{pfa}(x,E)$, for all $x \in X$ and $E \in \Sigma_d$:*

$$P(x,E) = P_{ca}(x,E) + P_{pfa}(x,E),$$

where $P_{ca}(x,\cdot) \in ca(X,\Sigma_d), P_{pfa}(x,\cdot) \in pfa(X,\Sigma_d)$, and $P_{ca}(\cdot,E) \in B(X,\Sigma_d)$, $P_{pfa}(\cdot,E) \in B(X,\Sigma_d)$.

Proposition 1 makes it possible to introduce integral sub-Markov operators A_{ca} and A_{pfa} generated by the corresponding measurable subkernels and $A = A_{ca} + A_{pfa}$.

Definition 3 *We call a finitely additive MC on an arbitrary discrete space (X,Σ_d) combined if its transition functionsatisfies the conditions: $P_{ca}(x,X) = q_1, P_{pfa}(x,X) = q_2$ for all $x \in X$, where $0 \leq q_1, q_2 \leq 1$, $q_1 + q_2 = 1$.*

2 Main results

Everywhere below we consider a combined non-degenerate finitely additive MC on an arbitrary discrete space (X,Σ_d).

Let there be given an arbitrary initial probability measure $\mu^1 \in S_{ba}$, $\mu^1 = \mu^1_{ca} + \mu^1_{pfa}$.

Take the second iteration in the Markov sequence of measures $\mu^2 = A\mu^1$. Then

$$\mu^2 = \mu^2_{ca} + \mu^2_{pfa} = A\mu^1 = (A_{ca} + A_{pfa})(\mu^1_{ca} + \mu^1_{pfa}) \\ = A_{ca}\mu^1_{ca} + A_{ca}\mu^1_{pfa} + A_{pfa}\mu^1_{ca} + A_{pfa}\mu^1_{pfa}. \quad (1)$$

In the last four terms of the decomposition (1), the first is a countably additive measure, the third and fourth are purely finitely additive measures.

The second term $A_{ca}\mu^1_{pfa}$ can be a measure of any type. Consider two corresponding main cases - disjoint conditions (H_1) and (H_2).

$$(H_1) \qquad A_{ca}(V_{pfa}) \subset V_{ca},$$

that is, the operator A_{ca} transforms all purely finitely additive measures from V_{pfa} into countably additive measures. MCs satisfying this condition (H_1) exist.

Theorem 1 *Let condition (H_1) be satisfied for some MC. Then for any initial measure $\mu^1 \in S_{ba}$, for any $n \in N$,*

$$\|\mu^{n+1}_{ca}\| = q_1, \quad \|\mu^{n+1}_{pfa}\| = q_2.$$

We now give the second condition (H_2) related to the decomposition in (1).

$$(H_2) \qquad A_{ca}(V_{pfa}) \subset V_{pfa},$$

that is, the operator A_{ca} transforms all purely finitely additive measures from V_{pfa} into purely finitely additive measures. Such MC exist.

Theorem 2 *Let condition (H_2) be satisfied for some MC. Then, for any initial finitely additive measure $\mu^1 \in S_{ba}$, for any $n \in N$*

$$\|\mu_{ca}^{n+1}\| = q_1^n \cdot \|\mu_{ca}^1\|, \quad \|\mu_{pfa}^{n+1}\| = 1 - q_1^n \cdot \|\mu_{ca}^1\|.$$

Corollary 1 *Let the conditions of Theorem 2 be satisfied. Then for any finitely additive initial measure $\mu^1 \in S_{ba}$ for the components of the Markov sequence of measures generated by it $\mu^{n+1} = A\mu^n$ as $n \to \infty$,*

$$\|\mu_{ca}^n\| \to 0 \text{ and } \|\mu_{pfa}^n\| \to 1.$$

Moreover, the convergence is uniform with respect to the initial measures $\mu^1 \in S_{ba}$ and exponentially fast.

It is desirable to find simple analogues of these conditions (H_1) and (H_2) in terms of the properties of the transition functions considered by the MC. We offer two such conditions.

$$(G_1) \begin{cases} \text{There is a finite set } D \subset X \text{ such that for all } x \in X: \\ P_{ca}(x, D) = P_{ca}(x, X) = q_1, \text{ which is equivalent to } P_{ca}(x, X \setminus D) = 0. \end{cases}$$

Theorem 3 *Let condition (G_1) be satisfied for some MC. Then 1) the condition (H_1) is satisfied, 2) the assertion of Theorem 1 is true.*

Consider one more condition (G_2) on the transition function of the MC. For an arbitrary $y \in X$ we denote the set $Q_y = \{x \in X : P_{ca}(x, \{y\}) > 0\}$.

$$(G_2) \qquad \text{For any } y \in X \text{ the set } Q_y \text{ is empty or finite.}$$

Theorem 4 *Let condition (G_2) be satisfied for some MC. Then 1) the condition (H_2) is satisfied, 2) the assertion of Theorem 2 is true.*

References

[1] S. Ramakrishnan, *Finitely Additive Markov Chains*, Trans. Amer. Math. Soc., **265**:1 (1981), 247–272.

[2] A.I. Zhdanok, *Finitely additive measures in the ergodic theory of Markov chains I*, Siberian Advances in Mathematics, **13**:1 (2003), 87–125; in Russian: A.I. Zhdanok, *Konechno-additivnyye mery v ergodicheskoy teorii tsepey Markova I*, Matematicheskiye trudy, **4**:2 (2001), 53–95.

[3] A.I. Zhdanok, *Finitely additive measures in the ergodic theory of Markov chains II*, Siberian Advances in Mathematics, **13**:2 (2003), 108–125; in Russian: A.I. Zhdanok, *Konechno-additivnyye mery v ergodicheskoy teorii tsepey Markova II*, Matematicheskiye trudy, **5**:1 (2002), 45–65.

[4] Zhdanok, A.I. *Cycles in Spaces of Finitely Additive Measures of General Markov Chains*, In: Recent Developments in Stochastic Methods and Applications. Springer Proceedings in Mathematics Statistics, Springer: New York, NY, USA, (2021), 131-143.