Growth optimal strategies in a large market

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In this work, we consider a model of a financial market which consists of a large agent ("the market") and a small agent (an individual investor), who invest money in dividend paying stock. Stock prices are determined by the actions of the large agent, and the small agent is a price-taker. The goal of the work is to find a strategy of the large agent which does not allow a small agent to achieve long-term growth of wealth greater than that of the large agent. The motivation for studying this problem arises from the known empirical fact that it is not possible "to beat" the market in the long run. If one assumes that this fact is true, it can be used to describe long-term behavior of the market. A related model was considered by Kardaras [1], but his setting does not lead to a single optimal strategy.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{P})$ be a filtered probability space satisfying the usual assumptions and the filtration having the property that any martingale has a continuous modification.

There are N assets in the market which pay dividends with intensities X_t^n per unit of time, $n = 1, \dots, N$. The supply of each asset (the number of shares in circulation) is normalized to 1. The dividends are paid in some perishable good and must be consumed by the agents immediately; there is no possibility to store the good. The intensity processes X_t^n are non-negative càdlàg semimartingales satisfying some non-degeneracy conditions.

There are two agents in the market, "large" and "small", who have wealth processes W_t and w_t. They are interpreted as "the market agent" who holds the whole market wealth, and an individual investor who has infinitesimal wealth (a rigor interpretation can be given by considering a the model with the number of agents going to infinity).

The agents' investment strategies are identified with processes $\Lambda_t = (\Lambda_t^1, \dots, \Lambda_t^N)$ and $\lambda_t = (\lambda_t^1, \dots, \lambda_t^N)$ with values in the N-simplex $\Delta_N = \{x \in \mathbb{R}_+^N : x^1 + \ldots + x^N = 1\}$. These processes show in which proportions the agents divide their wealth for investment in the assets. Both agents have the same consumption rate $\rho > 0$.

The model assumes that the wealth dynamics is defined by the equations

$$W_t = \frac{1}{\rho} \sum_{n=1}^{N} X_t^n, \tag{1}$$

$$dw_t = \sum_{n=1}^{N} \frac{\lambda_t^n w_t}{S_t^n} (dS_t^n + X_t^n dt) - \rho w_t dt, \tag{2}$$

where S_t^n are the stock prices, which are defined by the relation

$$S_t^n = \Lambda_t^n W_t. \tag{3}$$

Equation (1) follows from that the dividends must be fully consumed and the total market wealth is held by the large agent. Consequently, the strategy of this agent determines the asset prices: we assume that the supply of each asset is 1, so the price is equal to the amount of money invested in the asset, which gives equation (3). Equation (2) is the standard self-financing condition.

Definition 1. We say that the small agent *does not beat the market* in the long run if

$$\limsup_{t\to\ 0} \frac{w_t}{W_t} < \infty \text{ a.s.}$$

A strategy of the large agent is called *market growth optimal* if it cannot be beaten by any strategy of the small agent.

The main result of the paper consists in construction of a market growth optimal strategy.

Theorem 1. *The strategy $\widehat{\Lambda}_t = (\widehat{\Lambda}_t^1, \ldots, \widehat{\Lambda}_t^N)$ with the components*

$$\widehat{\Lambda}_t^n = e^{\rho t}\, \mathrm{E}\biggl(\int_t^\infty \rho e^{-\rho s} \frac{X_s^n}{\sum_{i=1}^n X_s^i} ds \biggm| \mathcal{F}_t \biggr)$$

is market growth optimal.

References

[1] Kardaras, C. (2008). Balance, growth and diversity of financial markets. *Annals of Finance*, 4(3), 369-397.