Nasyrov F. S. (Ufa, Russia) **The maximum principle for stochastic differential equations with a path-wise cost functional.**

Let $(\Omega, \mathcal{F}, (\mathcal{H}_t)_{t\in[0,T]}, P)$ be a complete filtered probability space. Suppose the filtration (H_t) is generated by the random process $V(t)$, $t \in [0,T]$, with continuous sample path. Lets introduce *a pathwise cost functional* $J(u(.)) = \int_0^T f_0(t, x(t), u(t))\,dt + h_0(T, x(0), x(T))$, here $x(t)$ is a solution of a controlled stochastic differential equation $x(t) = x_0 + \int_0^t b(s, u(s), x(s))ds + \int_0^t \sigma(s, x(s)) * dV(s)$, $t \in [0,T]$, and the second integral is a symmetric integral with respect to $V(t)$. By $\mathcal{U}$ we mean *piecewise-continuous H_t-adapted controls* $u(.)$ subject to standard form constraints $u(s) \in U_s$. Suppose T and x_0 are fixed.
So we have a problem: minimize $J(u(.))$ over $\mathcal{U}$.

Theorem. *Let the functions $b(t,x,u)$, $\sigma(t,x)$, $f_0(t,x,u)$, $h_0(T,x(0),y)$ be twice continuously differentiable functions with respect to all their variables, $\sigma(t,x) \neq 0$ for all (t,x). Suppose that the equation $(h_0)'_x(T, x(0), x) = 0$ is uniquely solvable with respect to x.*

Let the triplet $(x(s) = \varphi(s, V(s)),\ \psi(s) = \theta(s, x(s)),\ \Lambda(s, x(s)))$ be a solution of system

$$\begin{cases} dx(s) = b(s, x(s), u(s))ds + \sigma(s, x(s)) * dV(s), \quad x(0) = x_0, \\ d\psi(s) = [(f_0)'_x(s, x(s), u(s)) - \psi(s) b'_x(s, x(s), u(s))]ds \\ \qquad + [\Lambda'_x(s, x(s)) - \psi(s)\sigma'_x(s, x(s))] * dV(s), \\ \psi(T) = \Lambda'_x(T, x(T))/\sigma(T, x(T)) - (h_0)'_x(T, x(0), x(T)),\ s \in [0,T], \end{cases}$$

where the function $u(s) = u(s, x(s), \psi(s))$ is found from the maximum principle $H(s, x, \psi, u(s, x, \psi)) = \max_{u\in U(s)} H(s, x, \psi, u)$, where $H(s, x, \psi, u) = \psi b(s, x, u) - f_0(s, x, u)$. Then the system is a boundary value problem of the maximum principle for problem, that is, the triplet $(x(s), \psi(s), \Lambda(s, x()))$ satisfies the necessary conditions of the maximum principle for this problem.

The functions $x(s)$, $\psi(s)$ and $\Lambda(s, x)$ can be found from the system using the method proposed in this paper.

REFERENCES

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