Azarina S.V. A formula for the second order backward mean derivaves

Let us consider stochastic processes $\xi(t)$, $\eta(t)$ on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with values in $\mathbb{R}^n$. These processes are L_1 variables for all $t \in [0, T]$. We denote by E_t^η the conditional expectation of the process $\eta(t)$ for the σ-algebra, generated by preimages of the borel sets for the $\eta(t) : \Omega \to \mathbb{R}^n$.

We call the backward mean derivative of the process $\xi(t)$ for the moment t the L_1 random variable $D_*^\eta \xi(t) = \lim\limits_{\triangle t \to +0} E_t^\eta \left(\frac{\xi(t) - \xi(t - \triangle t)}{\triangle t} \right)$.

These derivative can be considered as the composition of the $\xi(t)$ and the borel vector field $Z(t, x)$: $Z(t, x) = \lim\limits_{\triangle t \to +0} E \left(\frac{\xi(t) - \xi(t - \triangle t)}{\triangle t} \middle| \eta(t) = x \right)$.

Let $Z(t, x)$ be a C^2 vector field of $\mathbb{R}^n$. The backward mean derivative of $Z(t, x)$ along $\xi(t)$ is the L_1-random process $D_*^\eta Z(t, \xi(t)) = \lim\limits_{\triangle t \to +0} E_t^\eta \left(\frac{Z(t, \xi(t)) - Z(t - \triangle t, \xi(t - \triangle t))}{\triangle t} \right)$.

Consider a Wiener process $w(t)$ in $\mathbb{R}^n$ as a process $\eta(t)$ and consider $\xi(t)$ as a solution of the stochastic differential Itô equation $\mathrm{d}\xi(t) = a(t, \xi(t))\mathrm{d}t + A(t, \xi(t))\mathrm{d}w(t)$, where the drift $a(t, x)$ and diffusion summand $A(t, x)$ are the vector field and the field of linear operators in $\mathbb{R}^n$ respectively.

We can prove the following statement using the backward Itô-Ventzell formula.

Theorem 1. We can compute the second order backward mean derivative $D_*^w D_*^w \xi(t)$ for the drift equals to 0, $a(t)$ and $a(t, x)$ and diffusion summand equals to A, Aw, $A(t)$.