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On Some Extensions of Lorden's Inequality

The classical Lorden inequality (1970) gives an upper bound for $\sup_{t\geq 0}\mathbb{E}R_t$, where $R_t = S_{N(t)} - t$ is the overshoot of a one-dimensional random walk over the level t.

Let $(Z_k)_{k\geq 1}$ be independent identically distributed random vectors in $\mathbb{R}^d$ with non-negative coordinates. Fix a norm $\|\cdot\|$ and set

$$S_0 = 0, \qquad S_n = \sum_{k=1}^{n} Z_k, \qquad N(t) = \inf\{n \geq 1 : \|S_n\| > t\}, \qquad R_t = \|S_{N(t)}\| - t.$$

Assume that

$$\mathbb{E}\|Z_1\|_1 > 0, \qquad \mathbb{E}\|Z_1\|_1^2 < \infty, \qquad \mathbb{E}\|Z_1\|^2 < \infty.$$

Theorem 1. Suppose there exists $C > 0$ such that $C\|x\|_1 \leq \|x\|$ for every $x \in \mathbb{R}^d$. Then

$$\sup_{t\geq 0}\mathbb{E}R_t \;\leq\; \left(\tfrac{1}{C} + \tfrac{1}{\|\mathbf{1}\|}\right)\frac{\mathbb{E}\|Z_1\|^2 + \mathbb{E}\|Z_1\|_1^2}{\mathbb{E}\|Z_1\|_1}.$$

Theorem 2. Let $\|\cdot\| = \|\cdot\|_2$ and $d \geq 1$. Then

$$\sup_{t\geq 0}\mathbb{E}R_t \;\leq\; 2\Big(\sqrt{d} + \tfrac{1}{\sqrt{d}}\Big)\,\frac{\mathbb{E}\|Z_1\|_1^2}{\mathbb{E}\|Z_1\|_1}.$$

The estimate in Theorem 2 is asymptotically optimal in its order with respect to the dimension d and, for $d = 1$, reduces to Lorden's classical result. Because the function $t \mapsto \mathbb{E}R_t$ is sub-additive, the Law of Large Numbers yields the asymptotic behaviour

$$\frac{\mathbb{E}R_t}{t} \;\leq\; \left(\frac{\mathbb{E}\|Z_1\|}{\|\mathbb{E}Z_1\|} - 1\right) + o(1), \qquad t \to \infty.$$

The inequalities are useful, for instance, in portfolio-risk assessment. Passing from a scalar risk measure to a vector of returns $(Z_k^{(1)}, \ldots, Z_k^{(d)})$ naturally suggests the use of the ℓ^1-norm. Theorem 2 provides an upper guarantee for the portfolio draw-down under a price shock of size t.