Nasyrov F. S. (Ufa, Russia) On the maximum principle with a path-wise cost functional for one-dimensional stochastic differential equations.

Let $(\Omega, \mathcal{F}, (\mathcal{H}_t)_{t\in[0,T]}, P)$ be a filtered probability space, where (H_t) is generated by a random process $V(t)$, $t \in [0,T]$, with continuous trajectories. We introduce *pathwise cost functional* $J(u(.)) = \int_0^T f_0(t, x(t), u(t))\, dt + h_0(T, x(0), x(T))$, where $x(t)$ is the solution of the controlled SDE

$$x(t){=}x_0 + \int_0^t b(s, u(s), x(s))ds{+} \int_0^t \sigma(s, u(s), x(s)) * dV(s),\ t \in [0,T], \tag{1}$$

here the second integral on the right-hand side is (see [4]) the symmetric integral over the process $V(t)$. Denote by $\mathcal{U}$ *piecewise continuous H_t-non-anticipatory controls* $u(.)$.

Consider the problem of minimizing the functional $J(u(.))$ in the class of non-anticipatory piecewise continuous controls $u(s) \in U$ under constraints of the form (1).

Theorem. *Let the functions $b(t,x,u)$, $\sigma(t,x,u)$, $f_0(t,x,u)$, $h_0(T,x(0),y)$ be twice continuously differentiable with respect to all their variables, the function $\sigma(t,x,u) \neq 0$ for all (t,x,u). Let the pair $(x(s), \psi(s))$ be a solution to the system*

$$\begin{cases} dx(s) = b(s, x(s), u(s))ds + \sigma(s, x(s), u(s)) * dV_(s), \\ d\psi(s){=}[(f_0)'_x(s, x(s), u(s)){-}\psi(s)b'_x(s, x(s), u(s))]ds \\ +[\Lambda'_x(s, x(s), u(s)) - \psi(s)\sigma'_x(s, x(s), u(s))] * dV(s), \\ x(0) = x_0, \quad \psi(T) = -h'_x(T, x(0), x(T)),\ s \in [0,T], \end{cases} \tag{2}$$

where the function $u(s){=}u^*(s,x,\psi)$ is found from the maximum condition $H^{(1)}(s,x,\psi,u^*(s,x,\psi)){=}\max_{u\in U} H^{(1)}(s,x,\psi,u)$, $H^{(1)}(s,x,\psi,u) = \psi b(s,x,u) - f_0(s,x,u)$, and the function $\Lambda(T,x,u(T))$ is selected so that $\max_{u\in U} \psi\sigma(s,x,u) - \Lambda(s,x,u)$ is achieved at the point $u = u^*(s,x(s),\psi(s))$. Then system (2) is a boundary value problem of the maximum principle for the problem under study, i.e. the pair $(x(s),\psi(s))$ satisfies the necessary conditions of the maximum principle.

The functions $x(s)$, $\psi(s)$ can be found from system (2) by the method proposed by the author in this work. The essence of the method is that under certain assumptions, first the solution of the second equation of system (2) is reduced to solving a first-order partial differential equation, and then the first equation is reduced to solving a first-order ODE chain.

This problem was previously solved (see [2], [3]) under the assumption that the control in equation (1) affects only the first term. Let us emphasize the fact (see [1]) that the solution of the problem: *to minimize $E\{J(u(.))\}$ in the class $\mathcal{U}$*, was reduced to solving two $FBSD$-systems of equations, which significantly complicated its application.

REFERENCES

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