On stochastic non-linear discounting, Presman Ernst (*CEMI RAS*), Zhang Yi (*University of Birmingham*). We consider the problem of nonlinear discounting in a stochastic formulation and show that under some natural assumptions it reduces to a standard stochastic optimal control problem with minimizing total costs.

Let $h = \{x_i\}_{i=0}^{\infty}$ be a sequence of some elements of Borel space X, $u(x_i)$ defines a current cost (utility) of x_i, and $w(x_i)$ defines a final cost (utility). Let $h_{r,n} = \{x_i\}_{i=r}^{n}$. One can consider as the cost (utility) of the set $h_{0,n}$ the following values

$$v_n(h_{0,n}) = u(x_0) + \gamma(u(x_1) + \gamma(u(x_2) + \cdots + \gamma(u(x_{n-1}) + \gamma(w(x_n))) \cdots)). \qquad (1)$$

with some increasing function $\gamma(\cdot)$. If $\gamma(u) = ku$ we have a standard discounted total cost with a discount coefficient k, $0 < k < 1$. As far as we know, first time (1) was proposed in [1] for the case $w(x) \equiv u(x)$.

Now let h be a sequence of random elements, $\{\mathcal{F}_i\}_{i=0}^{\infty}$ — a filtration of σ-algebras, such that $\mathcal{F}_i \subseteq \sigma(h_{0,i})$, $\mathcal{F}_i$ does not belong to $\sigma(h_{0,i-1})$. In [1]–[3] as the nonlinear discounted cost for fixed n was considered the value $U_n = U_{n,n}$, which was defined sequentially as follows

$$U_{n,0} = E\{w(x_n) \,|\, \mathcal{F}_{n-1}\}, \qquad U_{n,r} = E\{[u(x_{n-r}) + \gamma(U_{n,r-1})] \,|\, \mathcal{F}_{n-r-1}\} \text{ for } 1 \le r \le n, \quad (2)$$

where for convenience we assume that $\mathcal{F}_{-1}$ consists of the entire space and the empty set. If h is deterministic then $U_n = v_n(h_{0,n})$.

Let $u(x) > 0, w(x) > 0$, and the strictly increasing function $\gamma(u)$ is non-negative, concave, and $\gamma(u) - \gamma(0) < u$ for $u > 0$ (for example, $\gamma(u) = \ln(1+u)$). We will suppose also, that $\max[E\{u(x_i)\}, E\{w(x_i)\} < \infty$ for any $i \ge 0$. Let $B = \{b = (k, \delta) : k(u+\delta) \ge \gamma(u)$ for any $u \ge 0\}$. The sequence of pairs of functions $\mu = \{\mu_i = (\kappa_i,\ \delta_i)\}_{i=0}^{\infty}$, taking values in B and such that μ_i is measurable with respect to $\mathcal{F}_i$, $i \ge 0$, will be called the player's strategy. The set of all possible strategies will be denoted by $\mathcal{B}$.

Let us suppose that we have now a player who can at each time i either (with probability κ_i) to stop observation and the current cost at all subsequent moments of time will be equal to zero, or continue, and the cost at time $i+1$ will be $\tilde{u}(x_{i+1}) = u(x_{i+1}) + \delta_i$. Let τ be coresponding stopping time, $z_i = x_i$ if $i \le \tau$, $z(x_i) = \Delta$ if $i > \tau$, $\tilde{u}(\Delta) = 0$, E^{μ} fi the mathematical e[pectation generated by the strategy μ.

Theorem 1. *For any $n > 0$ there exists a strategy $\bar{\mu}^{(n)} = (\bar{\mu}_0^{(n)}, \ldots, \bar{\mu}_{n-1}^{(n)})$ such that*

$$U_n = \inf_{\mu \in \mathcal{B}} U_n^{\mu} = U_n^{\bar{\mu}^{(n)}} = E^{\bar{\mu}^{(n)}} \left\{ u(z_0) + \sum_{i=1}^{n-1} \tilde{u}(z_i) + \tilde{w}(z_n) \right\}. \qquad (3)$$

Theorem 2. *If $\liminf_{n\to\infty} U_n < \infty$ and $w(x) \le u(x)$, than there exists $U = \lim_{n\to\infty} U_n < \infty$, which does not depend on $w(\cdot)$, and the strategies $\bar{\mu}^{(n)}$ can be chosen in such a way that for any $i \ge 0$ there exists $\lim_{n\to\infty} \bar{\mu}_i^{(n)} = \bar{\mu}_i$, and for the limit strategy $\bar{\mu}$ the equality holds*

$$U = \inf_{\mu \in \mathcal{B}} E^{\mu} \left\{ u(z_0) + \sum_{i=1}^{\infty} \tilde{u}(z_i) \right\} = E^{\bar{\mu}} \left\{ u(z_0) + \sum_{i=1}^{\infty} \tilde{u}(z_i) \right\}. \qquad (4)$$

References: 1. Jaśkiewicz, A., Matkowski, J. and Nowak, A. *Math. Oper. Res.* Vol. 38, 2013, 108–121; 2. Bäuerle, N., Jaśkiewicz, A., and Nowak, A. *Appl. Math. Optim.* Vol 84, 2819–2848; 3. Piunovskiy, A., Presman E.and others, *Annals. of Oper. Res.* 2025, open access 11 March 2025.