Modelling the turnover of portfolio of hedge fund alphas via covariance matrix of their returns.

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A hedge fund *alpha* is an algorithmic trading strategy which generates a vector of positions in trading assets by the predefined moments of time. Alpha positions are both long and short; the sum of all the positions is zero and the sum of absolute values of the positions is 1. Alpha (or portfolio) *turnover* is l_1 norm of difference of position vectors calculated at consecutive time moments. The trades of opposite signs (buy/sell) *cross* each other as alphas are linearly combined in a portfolio. Then the portfolio turnover becomes a non linear function of alpha turnovers. Kakushadze and Liew (in [1]) and Kakushadze (in [2]) proposed estimations of the portfolio turnover via alpha weights, turnovers and covariance matrix of returns. However, the estimations were given without completely-defined mathematical model, proofs and conditions when they may be applicable. The aim of our joint with A.V. Kuliga work ([3]) was to fix this issue, to investigate the question on the mathematical level of rigor.

Let us call n–dimensional ($n \geq 2$) random vector $\alpha = (\alpha_1, \dots, \alpha_n)$ on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ *admissible*, if the variance $\mathrm{D}\alpha_i = 1$ for all $i = 1, \dots, n$ and non-zero linear combination of vector components cannot be equal to a constant almost surely. By $V(\alpha)$ we denote the linear space of linear combinations $\sum_{i=1}^{n} x_i\alpha_i$, where $x_i \in \mathbb{R}$ and α is an admissible vector. A function $f : V \to \mathbb{R}$ defined on the linear space V is called *absolutely homogeneous* of degree 1 if $f(\lambda v) = |\lambda| f(v)$ for all $\lambda \in \mathbb{R}$ and $v \in V$. So turnover is an absolutely homogeneous function on $V(\alpha)$. By C and D we denote the covariance matrix of several and the variance of one random variable correspondingly.

Theorem 1. *Let α be an admissible random vector and $f : V(\alpha) \to \mathbb{R}$ be an absolutely homogeneous function of degree 1. Then the existence of a function F such that*

$$f(\xi_1 + \xi_2) = F(C(\xi_1, \xi_2), f(\xi_1), f(\xi_2)) \quad \textit{for all} \quad \xi_1, \xi_2 \in V(\alpha),$$

is equivalent to the existence of a constant f_0 such that $f(\xi) = f_0\sqrt{\mathrm{D}(\xi)}$ for all $\xi \in V(\alpha)$.

Model correctness condition. If the portfolio turnover is expressed as a function of the covariance matrix of returns, weights and turnovers of alphas, then the ratio of turnover to the standard deviation of alpha returns should be the same for all the alphas.

References:

[1] Z. Kakushadze, J.K.-S. Liew. Is it possible to od on alpha? *The Journal of Alternative Investments*, 18(2): 39–49, (2015). https://arxiv.org/pdf/1404.0746

[2] Z. Kakushadze. Spectral model of turnover reduction. *Econometrics*, 3(3): 577–589, (2015). https://arxiv.org/pdf/1404.5050

[3] A.V. Kuliga, I.N. Shnurnikov. Turnover of investment portfolio via covariance matrix of returns. https://arxiv.org/pdf/2412.03305

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