Asymtotics of empirical spectral distributions of structured random matrices

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Let $\mathbf{X}^{(ij)}$, $i,j=1,\ldots,k$ a symmetric family of $n\times n$ symmetric real random matrices. We will assume that for any fixed pair of indices (i,j), $i,j=1,\ldots,k$, the elements of the matrix $\mathbf{X}^{(i,j)}$ are independent (with respect to symmetry), have mean zero and finite variance, i.e.

$$\mathbb{E}X_{pq}^{(i,j)}=0 \text{ and } \mathbb{E}(X_{pq}^{(i,j)})^2=(\sigma_{pq}^{(i,j)})^2<\infty.$$

About the matrices $\mathbf{X}^{(i,j)}$ we will assume that they either coincide at $(i,j)\neq(i_1,j_1)$ or are independent. Let the symbol $\otimes$ denote the Kronecker product.

Consider a matrix of the form $\mathbf{W_X}=\sum_{i,j=1}^k\mathbf{E}^{(i,j)}\otimes\mathbf{X}^{(i,j)}$, where $\mathbf{E}^{(i,j)}$ is a $k\times k$ matrix with all elements equal to zero except the (i,j)-th element equal to 1.

Let random variables $Y_{pq}^{(ij)}$ for any fixed $i,j=1<\ldots k, i\leq j$ be independent for $p,q=1,\ldots,n,p\leq q$, and have the same first two moments as random variables $X_{pq}^{(ij)}$. Let $\mathbf{W_Y}=\sum_{i,j=1}^k\mathbf{E}^{(i,j)}\otimes\mathbf{Y}^{(i,j)}$. We denote by $F_{n\mathbf{X}}(x)$ and $F_{n\mathbf{Y}}(x)$ the empirical spectral distribution functions of matrix $\mathbf{W_X}$ and $\mathbf{W_Y}$ respectively.

Theorem 1. *We assume that there exists a constant C_0 such that for all $n\geq1$*

$$\frac{1}{(nk)^2}\sum_{i,j=1}^k\sum_{p,q=1}^n(\sigma_{pq}^{(i,j)})^2\leq C_0.$$

Suppose that for all $i,j=1,\ldots,k$ for random matrices $\mathbf{X}^{(ij)}$ and $\mathbf{Y}^{(ij)}$ the Lindeberg condition is satisfied. i.e., for any $\tau>0$

$$L_n(\tau):=\max\{\frac{1}{n^2}\sum_{r,s=1}^n\mathbb{E}\big(X_{rs}^{(ij)}\big)^2\mathbb{I}\{|X_{rs}^{(ij)}|>\tau\sqrt{n}\},$$
$$\frac{1}{n^2}\sum_{r,s=1}^n\mathbb{E}\big(Y_{rs}^{(ij)}\big)^2\mathbb{I}\{|Y_{rs}^{(ij)}|>\tau\sqrt{n}\},\}\xrightarrow[n\to\infty]{}0.$$

Then

$$F_{n\mathbf{X}}(x)-F_{n\mathbf{Y}}(x)\xrightarrow[n\to\infty]{}0 \textit{ in probability.}$$