Systems of forward and backward nonlinear Kolmogorov equations

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Systems of forward and backward nonlinear parabolic equations arise as mathematical models of various phenomena in physics, chemistry, biology and many other fields. From the PDE point of view the main difference between forward and backward systems is the fact that commonly forward systems have to be treated as systems with respect to measures (or their densities) while the backward systems should be treated as systems of equations with respect to functions. From probabilistic point of view this means that systems of the first type correspond to systems of forward Kolmogorov equations [1], [2], while systems of the second type correspond to systems of backward Kolmogorov equations [3], [4]. In this talk we discuss probabilistic interpretation of several systems of forward Kolmogorov equations, namely the MHD-Burgers system and the Brusselator system as particular cases of systems of the form

$$\frac{\partial u_k}{\partial t} + div f(u) = \frac{\sigma_k^2}{2}\Delta u, \quad u_k(0,x) = u_{0k}(x), k = 1, 2, \dots d_1. \tag{1}$$

For generalised solutions of the Cauchy problem for these systems we derive corresponding probabilistic representations in terms of certain Markov processes and their multiplicative functionals. Moreover we show that one can reduce (1) to a closed stochastic system which can be applied to construct a numerical solution to (1). In particular for MHD-Burgers system, when $d_1 = 2$, $f(u) = \left(u_1 u_2, \frac{u_1^2+u_2^2}{2}\right)^*$, the corresponding stochastic system has the form

$$d\hat{\xi}_k(\theta) = -\sigma_k dw(\theta), \quad \hat{\xi}_k(0) = x, \quad k = 1, 2, \tag{2}$$

$$d\tilde{\eta}^k(\theta) = C_u^k(\hat{\xi}(\theta))\tilde{\eta}(\theta)dw(\theta), \quad \tilde{\eta}^k(0) = 1, \tag{3}$$

$$u^k(t,x) = E[\tilde{\eta}^k(t)u_{0k}(\hat{\xi}^k(t))]. \tag{4}$$

Theorem 1. *1. Assume that there exists a unique regular generalised solution to (1) corresponding the the MHD-Burgers system. Then this solution admits a stochastic representation of the form (4), where $\xi_k(\theta), \tilde{\eta}^k(\theta)$ satisfy (2),(3) and*

$$C_u^1(x) = \sigma^{-1}u_2(\theta,x), \quad C_u^2(x) = [2\mu]^{-1}\left(u_2(\theta,x) + u_1^2(\theta,x)[u_2(\theta,x)]^{-1}\right).$$

2. Let $u_k > 0$ and $\nabla u_k \in L^2$. Then there exists an interval $[0,T]$ such that for all $t \in [0,T]$ there exists a unique solution of the system (2)–(4). In addition the functions $u_k(s,x), k = 1, 2$ of the form (4) satisfy the Cauchy problem for the MHD-Burgers system the sense of Schwartz distribution theory.

We discuss as well a probabilistic interpretation of the Cauchy problem solutions for parabolic systems with cross-diffusion [3], [4].

References

[1] Belopolskaya, Ya. Stochastic interpretation of quasilinear parabolic systems with cross diffusion Theory of Probability and its Applications 61(2), 208–234 (2017).

[2] BelopolskayaYa., Stepanova A. Stochastic interpretation of MHD-Burgers system Za- piski nauchn.sem PDMI. 446 7–29 (2017).

[3] Belopolskaya Ya. Probabilistic counterparts for strongly coupled parabolic systems Springer Proceedings in Mathematics & Statistics. Topics in Statistical Simulation. v 114. P. 33-42.

[4] BelopolskayaYa., Probabilistic Models of the Conservation and Balance Laws in Switching Regimes, Journal of Mathematical Sciences, v 229 N 6 (2018) 601–625.

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