Ulyanov V.V. (Moscow, Russia) – **Non-asymptotic bounds for the closeness of Gaussian measures of the balls.**

The talk provides an overview of the tight non-asymptotic bounds for the Kolmogorov distance between the probabilities of two Gaussian elements to hit a ball in a Hilbert space. The key property of these bounds is that they are dimension-free and depend on the nuclear (Schatten-one) norm of the difference between the covariance operators of the Gaussian elements and on the norm of the mean shift. The obtained bounds significantly improve the bound based on Pinsker's inequality via the Kullback-Leibler divergence. We also establish an anti-concentration bound for a squared norm of a non-centered Gaussian element in a Hilbert space. We present a number of examples motivating our results and its applications to statistical inference and to high-dimensional CLT (see e.g. [1]). The formulations and proofs of the results mentioned in the talk can be found in [2–6], see also my pages on MathNet.ru and ResearchGate.net. Here we give simplified version of two key results from [4] and [5].

Let H be a real separable Hilbert space with norm $||\cdot||$.

Theorem 1 *Let ξ and η be Gaussian elements in H with zero mean and covariance operators Σ_ξ and Σ_η resp. Let $\lambda_{1\xi} \geq \lambda_{2\xi} \geq \ldots$ and $\lambda_{1\eta} \geq \lambda_{2\eta} \geq \ldots$ be the eigenvalues of Σ_ξ and Σ_η resp. Then there exists an absolute constant C such that for $\Lambda_{k\xi}^2 := \sum_{j=k}^{\infty} \lambda_{j\xi}^2$ and $\Lambda_{k\eta}^2 := \sum_{j=k}^{\infty} \lambda_{j\eta}^2$, $k = 1, 2$, one has:*

$$\sup_{x>0} |P(\|\xi\| \leq x) - P(\|\eta\| \leq x)| \leq C \left((\Lambda_{1\xi}\Lambda_{2\xi})^{-1/2} + (\Lambda_{1\eta}\Lambda_{2\eta})^{-1/2} \right) \sum_{i=1}^{\infty} |\lambda_{i\xi} - \lambda_{i\eta}|.$$

The following estimate for a probability density function $p(x)$ of $\|\xi\|^2$ plays an important role in the proof of Theorem 1.

Lemma 1 *Let ξ be a Gaussian element in H with zero mean and a covariance operator Σ_ξ. Then there exists an absolute constant c such that one has*

$$\max_{x \geq 0} p(x) \leq c\, (\Lambda_{1\xi}\Lambda_{2\xi})^{-1/2}. \quad (1)$$

Notice that if «effective» dimension of Σ_ξ is 2 at least, i.e. if $2\,\lambda_{1\xi}^2 \leq \Lambda_{1\xi}^2$, then $\Lambda_{1\xi} \approx \Lambda_{2\xi}$ and the right-hand side of (1) is inversely proportional to Frobenius norm $\Lambda_{1\xi}$ of Σ_ξ. In particular, in finite d-dimensional case $H = R^d$ for $d \geq 2$, if Σ_ξ is close to the identity matrix I then according to (1) one has $\max_{x\geq 0} p(x) \leq c\, d^{-1/2}$. This estimate is consistent with the maximum value of the chi-square distribution density function with d degrees of freedom.

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