Donchev D.S. (Sofia, Bulgaria) — **Asymptotic solutions to the inverse Shiryaev's problem.**

Let $B_t,\ t \geq 0$ be a Brownian motion. For any deterministic function $f(t),\ t \geq 0$ with $f(0) = y > 0$, which belongs to the class $C^1([0,\infty)) \cap C^2((0,\infty))$, we define and study the stopping time

$$\tau_f = \min\{t > 0 : B_t = f(t)\},\ \min(\phi) = \infty.$$

Let $p_f(t) = \frac{d}{dt}P(\tau_f \leq t),\ t > 0$ be the exit density of $B_t,\ t \geq 0$ through the boundary $f(t),\ t \geq 0$. In a recent paper (see [1]), we have shown that the exit density of a Brownian motion through one-sided boundaries is expressed as a solution of a parabolic second-order PDE. It turns out that this equation can be reduced to a first-order PDE whose solution admits an analytic representation only for three classes of boundaries - parabolic, square-root and rational. This result allows to make conclusions about global solutions to the first exit problem, i.e. representations of the density $p_f(t)$ on the positive half-line, only for a limited number of boundaries. However, it is still possible to look for asymptotic solutions. Indeed, we establish the following formula which is valid for small t:

$$p_f(t) = \frac{y}{2\sqrt{2\pi t^3}} e^{-\frac{y^2}{2t} - f'(0)y} \left(1 + e^{(f'(0) - f'(t))y} - \int_0^t (f'(u))^2 du + o(t)\right), \qquad (1)$$

provided that $f \in C^1([0,\infty)) \cap C^2((0,\infty))$.

A nice feature of this formula is that it connects explicitly both the exit density and the boundary. This fact allows to deal not only with the direct problem, but also with the inverse first exit problem. The inverse problem has been posed by A.N. Shiryaev in the 70s of last century as follows: for a given probability density function $q(t),\ t \geq 0$, find a boundary $f(t),\ t \geq 0$, such that $p_f(t) = q(t)$. Equivalently, we want to find a boundary $f(t),\ t \geq 0$ such that the exit time τ_f to "coincide" with a random variable, say η, whose density is $q(t),\ t \geq 0$.

Here, we show how formula (1) can help us to find small time asymptotic solutions to the inverse problem. Namely, we develop a procedure that allows us, for a given density $q(t),\ t > 0$ to construct a boundary $f(t),\ t \geq 0$ such that, for small t,

$$\log(p_f(t)) = \log(q(t)) + o(t).$$

REFERENCES

1. *Donchev D.S.* An excursion characterization of the first hitting time of a Brownian motion in a smooth boundary, Random Operators and Stochastic Equations, 2007, vol. 15, pp. 35–48.