Gushchin A.A. (Moscow, Russia) — **The joint law of the maximum and terminal value of a max-continuous local submartingale.**

In 1993, Rogers [1] described the class of all possible joint laws of the terminal value of a random process and its maximum for two families of processes, uniformly integrable martingales and a.s. convergent continuous local martingales vanishing at 0. It turns out that in the second case, it is possible and natural to extend the family of processes while preserving the corresponding class of joint laws. Namely, we consider the totality $\mathscr{X}$ of all a.s. convergent right-continuous local submartingales $X = (X_t)_{t\geqslant 0}$, $X_0 = 0$, such that their running maximum $\overline{X}_t := \sup_{s\leqslant t} X_s$ is continuous (such processes are sometimes called max-continuous). In the talk we discuss the following issues: 1) a description of the class of joint laws $\mathrm{Law}(X_\infty, \overline{X}_\infty)$ of the terminal value X_∞ of the process and its global maximum $\overline{X}_\infty$ when X runs over $\mathscr{X}$; 2) for every measure μ from this class to find one or another "simple" representative X from $\mathscr{X}$ with $\mathrm{Law}(X_\infty, \overline{X}_\infty) = \mu$. From Rogers' theorem it follows that there always exists a continuous local martingale with this property. We offer an alternative proof of this fact. The second question is interesting because, as we prove, whether the process X from $\mathscr{X}$ is a closed submartingale, or a closed supermartingale, or a uniformly integrable martingale depends only on the joint law $\mathrm{Law}(X_\infty, \overline{X}_\infty)$.

Proposition 1 *For a process X in $\mathscr{X}$, define a change of time by $C_s := \inf\{t : \overline{X}_t > s\}$. Then the time-changed process $Y := X \circ C := (X_{C_s})_{s\geqslant 0}$ is a max-continuous submartingale with respect to the filtration $(\mathscr{F}_{C_s})_{s\geqslant 0}$ and is expressed as*

$$Y_s = s\mathbb{1}_{\{s<\overline{X}_\infty\}} + X_\infty \mathbb{1}_{\{s\geqslant\overline{X}_\infty\}}.$$

In particular, $Y_\infty = X_\infty$ and $\overline{Y}_\infty = \overline{X}_\infty$.

Proposition 2 *Let W and L be random variables on some probability space $(\Omega, \mathscr{F}, \mathsf{P})$, $W \geqslant \max\{L, 0\}$ and the function $s \rightsquigarrow \mathsf{E}\big[s\mathbb{1}_{\{s<W\}} + L\mathbb{1}_{\{s\geqslant W\}}\big]$, $s \geqslant 0$, vanishes at 0 and is monotone increasing in s. Define $\mathscr{F}_s$ as the σ-algebra of subsets of $\mathscr{F}$ such that their intersection with $\{W > s\}$ is either empty or coincides with $\{W > s\}$. Put*

$$Y_s = s\mathbb{1}_{\{s<W\}} + L\mathbb{1}_{\{s\geqslant W\}}. \tag{1}$$

Then $Y = (Y_s)_{s\geqslant 0}$ is an $(\mathscr{F}_s)$-submartingale.

Propositions 1 and 2 provide solutions to the two issues stated above. Given a probability measure $\mu = \mu(dx, dy)$ on $\mathbb{R}^2$ wuth the support in the set $\{(x, y)\colon y \geqslant \max\{x, 0\}\}$, there is a process X in $\mathscr{X}$ such that $\mu = \mathrm{Law}(X_\infty, \overline{X}_\infty)$ if and only if the function

$$s \rightsquigarrow \int \big[s\mathbb{1}_{\{y>s\}} + x\mathbb{1}_{\{y\leqslant s\}}\big]\,\mu(dx, dy), \quad s \geqslant 0,$$

vanishes at 0 при $s = 0$ and is monotone increasing in s. For every measure μ with this property, we can construct a "simple" submartingale Y of form (1) such that $\mathrm{Law}(Y_\infty, \overline{Y}_\infty) = \mu$. If we now embed the submartingale Y into a Brownian motion by a change of time consisting of minimal stopping times using the Monroe theorem or its generalizations, then we can define in some way a continuous martingale X and justify that $\mathrm{Law}(X_\infty, \overline{X}_\infty) = \mu$.

REFERENCESr

1. *Rogers L.C.G.* The joint law of the maximum and terminal value of a martingale, Probab. Theory Relat. Fields, 1993, vol. 95, № 4, pp. 451–466.

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